

CHAOTIC AND TOPOLOGICAL PROPERTIES OF CONTINUED FRACTIONS

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ABSTRACT. We prove that there exists a scrambled set for the Gauss map with full Hausdorff dimension. Meanwhile, we also investigate the topological properties of the sets of points with dense or non-dense orbits.

1. INTRODUCTION

It is known that every irrational number $x \in [0, 1)$ can admit an infinite continued fraction (CF) induced by Gauss transformation $T : [0, 1) \rightarrow [0, 1)$ given by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } x \in (0, 1) \text{ and } T(0) := 0$$

where $\lfloor y \rfloor$ denotes the integer part of a real number y , that is, $\lfloor y \rfloor = n$, if $y \in [n, n+1)$ for some $n \in \mathbb{Z}$. We set

$$a_1(x) := \lfloor x^{-1} \rfloor \quad \text{and} \quad a_n(x) := \lfloor (T^{n-1}(x))^{-1} \rfloor, \quad n \geq 2,$$

and have the following CF expansion of x :

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}} = [a_1(x), a_2(x), a_3(x), \cdots].$$

The numbers $a_n(x)$ ($n \geq 1$) are called the partial quotients of x . In 1845, Gauss observed that T preserves the probability measure given by

$$\mu(B) = \frac{1}{\log 2} \int_B \frac{1}{1+x} d\mathcal{L}(x),$$

where $B \subset [0, 1)$ is any Borel measurable set and \mathcal{L} is the Lebesgue measure on $[0, 1]$. The measure μ is called Gauss measure and equivalent with \mathcal{L} .

Continued fractions is a kind of representation of real numbers and an important tool to study the Diophantine approximation in number theory. Many metric and dimensional results on Diophantine approximation have been obtained with the help of continued fractions such as Good [5], Jarnik [7], Khintchine [9] etc, and also the dimensional properties of continued fractions were considered, for example Wang and Wu [21], Xu [26] etc. The behaviors of the continued fraction dynamical systems have been widely investigated, for example, the shrinking target problems [13], mixing property [18], thermodynamic formalism [15], limit theorems [3] etc. Hu and Yu [6] were concerned with the set $\{x \in [0, 1) : \xi \notin \{T^n(x) : n \geq 0\}\}$ for any $\xi \in [0, 1)$ and proved that it is 1/2-winning and thus it has full Hausdorff dimension.

Asymptotic behaviors of the orbits are one of the main theme in dynamical systems. In this paper, we focus on the continued fraction dynamical system $([0, 1), T)$ and study firstly the denseness of the orbits, and secondly the size of the scrambled set from the sense of Hausdorff dimension. Let

$$D := \{x \in [0, 1) : \text{the orbit of } x \text{ under } T \text{ is dense in } [0, 1]\},$$

and D^c be the complement of the set D in $[0, 1]$.

Theorem 1.1. *Let T be the Gauss map on $[0, 1)$. Then*

- (1) *D is of full Lebesgue measure in $[0, 1]$.*
- (2) *D^c is of full Hausdorff dimension.*

Date: September 1, 2016.

2010 Mathematics Subject Classification. 11A55; 37F35; 74H65.

Key words and phrases. continued fractions, Hausdorff dimension, Li-York chaotic.

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- (3) D^c is of the first category in $[0, 1]$.
(4) D^c and D are dense in $[0, 1]$.

The second statement (2) of the theorem is a direct corollary of the result in [6] since the set studied in [6] is a subset of D^c . Similar properties with this theorem for β -transformations have been given by [12]. For some other dynamical systems, the set of the points with nondense orbits is usually of full Hausdorff dimension, see also [10], [20] and references therein.

The chaotic property is a characterization of the asymptotic behaviors between different orbits in dynamical system.

Definition 1.2. [14] Let (X, ρ) be a metric space. For two points $x, y \in X$, (x, y) is a scrambled pair for the map $f : X \rightarrow X$, if

$$\limsup_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) > 0 \text{ and } \liminf_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) = 0.$$

A subset $S \subseteq X$, containing at least two points, is a scrambled set of f , if for any $x, y \in S$, $x \neq y$, (x, y) is a scrambled pair for f . If a scrambled set S for f is uncountable, we say that f is chaotic in the sense of Li-Yorke.

It is well known that [1] the surjective continuous transformation on compact metric space with positive topological entropy is chaotic in the sense of Li-Yorke. Even though the topological entropy of the Gauss map is infinite, the result above in [1] can not be applied since the Gauss map is not continuous. We prove that the Gauss map is chaotic as the following, indeed, the scrambled set can be very large from the dimensional sense, not just uncountable.

Theorem 1.3. *Let T be the Gauss map on $[0, 1)$. Then there exists a scrambled set in $[0, 1)$ whose Hausdorff dimension is 1.*

As a consequence, we have

Corollary 1.4. The Gauss transformation on $[0, 1)$ is chaotic in the sense of Li-Yorke.

The sizes of the scrambled sets have been considered for many dynamical systems from the sense of the measure, dimension, and topology, see also [2], [16], [17], [22], [23] etc. In 1995, Xiong [25] proved that there exists a scrambled set of $\{1, 2, \dots, N\}^{\mathbb{N}}$ of full Hausdorff dimension.

The paper is organized as follows. In Section 2, we collect and establish some elementary properties of continued fractions which will be used later. Section 3 is devoted to proving Theorem 1.1 and Theorem 1.3 is proved in Section 4.

2. PRELIMINARIES

In this section, we present some elementary results in the theory of symbolic dynamics and continued fractions which will be used later, see [9].

2.1. Basic concepts. Let $\mathcal{A} = \{1, 2, \dots, N\}$ with $N \geq 2$ or $\mathcal{A} = \mathbb{N} = \{1, 2, \dots, n, \dots\}$. Denote the symbolic space of one-sided infinite sequences over \mathcal{A} by

$$\mathcal{A}^{\mathbb{N}} = \{x = (x_1, x_2, \dots) : x_i \in \mathcal{A}, \forall i \in \mathbb{N}\}.$$

The symbol x_i is called the i -th coordinate of x .

We assign the discrete topology to \mathcal{A} and the product topology to $\mathcal{A}^{\mathbb{N}}$. For $x, y \in \mathcal{A}^{\mathbb{N}}$, the distance d is defined by

$$d(x, y) = 2^{-i}, \text{ where } i = \inf\{j \geq 0 : x_{j+1} \neq y_{j+1}\}.$$

Denote by \mathcal{A}^n the set of all n -letter words and \mathcal{A}^* for the set of all finite words over \mathcal{A} . For any $u, v \in \mathcal{A}^*$, uv denotes the concatenation of u and v . The symbol “ $|\cdot|$ ” means the diameter, the length and the absolute value with respect to a set, a word, and a real number respectively. For $i, j \in \mathbb{N}$ with $i < j$, we write $x_i^j = x_i, x_{i+1}, \dots, x_j$.

The shift map $\sigma : \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$ is defined by

$$(\sigma(x))_i = x_{i+1}, \forall i \in \mathbb{N}.$$

2.2. Continued fractions. In this subsection, we collect and establish some elementary properties on continued fractions. We can refer to [9] for more details.

We denote by $p_n(x)/q_n(x) := [a_1(x), a_2(x), \dots, a_n(x)]$ the n -th convergent of x . With the conventions $p_{-1}(x) = 1, q_{-1}(x) = 0, p_0(x) = 0, q_0(x) = 1$, we have

$$p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \quad n \geq 0,$$

$$q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \geq 0.$$

For any $n \geq 1$ and $(a_1, \dots, a_n) \in \mathbb{N}^n$, let $q_n(a_1, \dots, a_n)$ be the denominator of finite continued fraction $[a_1, \dots, a_n]$. If there is no confusion, we write q_n instead of $q_n(a_1, \dots, a_n)$ for simplicity. For any $n \geq 1$ and $(a_1, \dots, a_n) \in \mathbb{N}^n$, we define the n -th basic intervals $I(a_1, \dots, a_n)$ by

$$I(a_1, \dots, a_n) = \{x \in [0, 1) : a_i(x) = a_i, 1 \leq i \leq n\}.$$

In the case of $n = 1$, $I(i) = \left[\frac{1}{i+1}, \frac{1}{i}\right)$ for $i \geq 1$. The following lemmata are well known.

Lemma 2.1. [9] *For any $n \geq 1$ and $(a_1, \dots, a_n) \in \mathbb{N}^n$, we have:*

- (i) $q_n(a_1, \dots, a_n) \geq 2^{\frac{n-1}{2}}$;
- (ii) $|I(a_1, \dots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}$.

Lemma 2.2. [24] *For any $n \geq 1$ and $1 \leq k \leq n$,*

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, \dots, a_n)}{q_{n-1}(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)} \leq a_k + 1.$$

For any $N \in \mathbb{N}$, let

$$E_N = \{x \in [0, 1) : 1 \leq a_n(x) \leq N, \text{ for any } n \geq 1\},$$

and

$$E = \{x \in [0, 1) : \sup_{n \geq 1} a_n(x) < +\infty\} = \bigcup_{N=1}^{\infty} E_N.$$

Jarnik [7] showed the following:

Lemma 2.3. *For any $N \geq 8$,*

$$1 - \frac{1}{N \log 2} \leq \dim_H E_N \leq 1 - \frac{1}{8N \log N},$$

and

$$\dim_H E = 1,$$

where “ \dim_H ” denotes the Hausdorff dimension.

Lemma 2.4. [6, Lemma 4.1] *Let $T : [0, 1) \rightarrow [0, 1)$ be the Gauss transformation. Then there exists a constant $\lambda \geq 1$ such that for any $u, v \in \mathbb{N}^*$,*

$$\lambda^{-1}|I(u)||I(v)| \leq |I(uv)| \leq \lambda|I(u)||I(v)|.$$

3. THE PROOF OF THEOREM 1.1

Lemma 3.1. *Let $[a_1(x), a_2(x), a_3(x), \dots]$ be the CF expansion of $x \in [0, 1)$. Then $x \in D^c$ if and only if there exists some word $u \in \mathbb{N}^*$ which does not appear in $(a_1(x), a_2(x), a_3(x), \dots)$.*

Proof. On the one hand, if there exists $u \in \mathbb{N}^*$ which does not appear in $(a_1(x), a_2(x), a_3(x), \dots)$, let y be the middle point of the interval $I(u)$, then $|y - T^n(x)| \geq |I(u)|/2$ for any $n \geq 0$. Hence $x \in D^c$ by the definition of the set D^c .

On the other hand, suppose that $x \in D^c$. When $x \in \mathbb{Q} \cap [0, 1)$, since the CF expansion of every rational number is finite, then there exists $k = k(x) \geq 1$ such that $x = [a_1(x), a_2(x), \dots, a_k(x)]$, thus the finite word $u := (a_1(x), a_2(x), \dots, a_k(x), 1) \in \mathbb{N}^{k+1}$ is the required. Now it left to check the case of $x \in \mathbb{Q}^c \cap [0, 1)$.

By contradiction, suppose that every $u \in \mathbb{N}^*$ appears in $(a_1(x), a_2(x), a_3(x), \dots)$, then, for any $y \in \mathbb{Q}^c \cap [0, 1)$ and any $n \geq 1$, the finite word $(a_1(y), a_2(y), \dots, a_n(y)) \in \mathbb{N}^n$ will appear in $(a_1(x), a_2(x), \dots)$. Hence there exists $m = m(y, n)$ such that $T^m(x) \in I(a_1(y), a_2(y), \dots, a_n(y))$, then we obtain

$$|y - T^m(x)| \leq |I(a_1(y), a_2(y), \dots, a_n(y))| \leq q_n(y)^{-2} \leq 2^{-(n-1)}$$

by (i) of Lemma 2.1, it follows that

$$\mathbb{Q}^c \cap [0, 1) \subset \overline{\{T^k(x) : k \geq 0\}},$$

therefore

$$[0, 1] = \overline{\mathbb{Q}^c \cap [0, 1)} \subset \overline{\{T^k(x) : k \geq 0\}}.$$

that is to say, $x \in D$ by the definition of the set D , which contradicts $x \in D^c$. Then the lemma holds. \square

Proof of Theorem 1.1:

(1) In view of Lemma 3.1, we have

$$(3.1) \quad D^c = \bigcup_{k \geq 1} \bigcup_{u \in \mathbb{N}^k} \{x \in [0, 1) : u \text{ does not appear in } (a_1(x), a_2(x), a_3(x), \dots)\} := \bigcup_{k \geq 1} \bigcup_{u \in \mathbb{N}^k} F_u.$$

Since the cardinality of \mathbb{N}^k is countable, we just need to prove $\mathcal{L}(F_u) = 0$, where \mathcal{L} is the Lebesgue measure on $[0, 1]$. The fact that T is ergodic with respect to the Gauss measure μ is equivalent to that

$$\mu\left(\bigcup_{n \geq 0} T^{-n}(B)\right) = 1$$

for any Borel measurable subset $B \subseteq [0, 1)$ of positive measure. Since $\mu(I(u)) > 0$, we have

$$\mu\left(\bigcup_{n \geq 0} T^{-n}(I(u))\right) = 1.$$

Since $\bigcup_{n \geq 0} T^{-n}(I(u)) \cap F_u = \emptyset$, the assertion $\mu(F_u) = 0$ follows, which implies $\mathcal{L}(F_u) = 0$ by the equivalence between the Lebesgue and Gauss measures.

(2) Since $E_N \subseteq D^c$ for any $N \geq 1$ by Lemma 3.1, we have $E \subseteq D^c \subseteq [0, 1)$. Hence $\dim_H D^c = \dim_H E = 1$ by Lemma 2.3.

(3) According to (3.1), we only need to prove that F_u is nowhere dense since \mathbb{N}^k is countable. Recall that the Gauss measure μ is an ergodic measure with respect to T . Let b and c be the endpoints of the interval $I(u)$ and $b < c$. Since the open interval $(b, c) \subset I(u)$, we have

$$\mu\left(\bigcup_{n \geq 0} T^{-n}(b, c)\right) = 1, \text{ and } \bigcup_{n \geq 0} T^{-n}(b, c) \cap F_u \subset \bigcup_{n \geq 0} T^{-n}(I(u)) \cap F_u = \emptyset.$$

Then $F_u \subset \left(\bigcup_{n \geq 0} T^{-n}(b, c)\right)^c$. Let $\overline{F_u}$ be the closure of F_u . Since $\bigcup_{n \geq 0} T^{-n}(b, c)$ is an open set, thus

$$\overline{F_u} \subset \left(\bigcup_{n \geq 0} T^{-n}(b, c)\right)^c \text{ and } \mu(\overline{F_u}) \leq \mu\left\{\left(\bigcup_{n \geq 0} T^{-n}(b, c)\right)^c\right\} = 0.$$

Then $\mathcal{L}(\overline{F_u}) = 0$ by the equivalence between the Lebesgue and Gauss measures. Hence F_u is nowhere dense.

(4) Since $\mathcal{L}(D) = 1$ by (1), we know that D is dense in $[0, 1]$. It suffices to prove that D^c is dense in $[0, 1]$, which can be obtained by the facts that $E \subseteq D^c$ and that the set E is dense in $[0, 1]$. \square

4. THE PROOF OF THEOREM 1.3

The idea of proving Theorem 1.3 is to construct a scrambled set in $\mathbb{N}^{\mathbb{N}}$, then project such set to the unit interval $[0, 1)$, and finally show the projection is of full Hausdorff dimension.

4.1. Construction of a scrambled set. Fix $N \geq 2$, we define four mappings g_N , Θ_N , Ψ_N and Δ_N which are dependent on the symbol N . Actually, the idea for constructing these mappings is inspired by Xiong [25]. In the following, we write $\Sigma_N = \{1, 2, \dots, N\}^{\mathbb{N}}$ to emphasize the dependence of the number N of letters.

(1). Define a map $g_N : \Sigma_N \rightarrow \Sigma_N$ by

$$(g_N(x))_n = \begin{cases} N & \text{if } n = 1, \\ 1 & \text{if } 2 + k(k-1) \leq n \leq 1 + k^2 \text{ for } k \geq 1, \\ x_{n-1-k^2} & \text{if } 2 + k^2 \leq n \leq 1 + k^2 + k \text{ for } k \geq 1, \end{cases}$$

for $x = (x_1, x_2, x_3, \dots) \in \Sigma_N$. That is to say, the form of $g_N(x)$ is as follows:

$$g_N(x) = (N, 1, x_1, 1, 1, x_1, x_2, 1, 1, 1, x_1, x_2, x_3, \dots, \underbrace{1, 1, \dots, 1}_n, x_1, x_2, \dots, x_n, \dots)$$

(2). Define $\Theta_N : \Sigma_N \rightarrow \Sigma_N$ by

$$\Theta_N(x) = (x_1, x_1, x_2, x_1, x_2, x_3, \dots, x_1, x_2, \dots, x_n, x_1, x_2, \dots, x_{n+1}, \dots)$$

for $x = (x_1, x_2, x_3, \dots) \in \Sigma_N$.

(3). Define a positive integers sequence $R = \{r_n\}_{n \geq 1} := \bigcup_{m=1}^{\infty} \bigcup_{t=1}^m \{k : k = m^3 + t\}$, that is, $R = \{2, 9, 10, 28, 29, 30, \dots\}$. Define $\Psi_N : \Sigma_N \times \Sigma_N \rightarrow \Sigma_N$ by

$$(\Psi_N(x, y))_n = \begin{cases} y_k & \text{if there exists } k \geq 1 \text{ such that } n = r_k, \\ x_{n-k+1} & \text{if there exists } k \geq 1 \text{ such that } r_{k-1} < n < r_k, \end{cases}$$

for $x = (x_1, x_2, \dots) \in \Sigma_N$ and $y = (y_1, y_2, \dots) \in \Sigma_N$.

(4). Define $\Delta_N : \Sigma_N \rightarrow \Sigma_N$ such that for $x \in \Sigma_N$,

$$\Delta_N(x) = \Psi_N(x, \Theta_N \circ g_N(x)).$$

That is to say, the form of $\Delta_N(x)$ is as follows:

$$\begin{aligned} \Delta_N(x) &= (x_1, (\Theta_N(g_N(x)))_1, x|_2^7, (\Theta_N(g_N(x)))|_2^3, x|_8^{24}, (\Theta_N(g_N(x)))|_4^6, x|_{25}^{58}, (\Theta_N(g_N(x)))|_7^{10}, \dots) \\ &= (x_1, (g_N(x))_1, x|_2^7, (g_N(x))|_1^2, x|_8^{24}, (g_N(x))|_1^3, x|_{25}^{58}, (g_N(x))|_1^4, \dots) \end{aligned}$$

Remark 4.1. (1). The mappings g_N , Θ_N and Ψ_N are continuous and injective. The mapping Δ_N is a continuous bijection from Σ_N to $\Delta_N(\Sigma_N)$.

(2). $(\Theta_N(g_N(x)))_1 = (\Theta_N(g_N(x)))_2 = (g_N(x))_1 = N$ and $(\Theta_N(g_N(x)))_3 = (g_N(x))_2 = 1$ for all $x \in \Sigma_N$.

Let $S_N := \Delta_N(\Sigma_N)$ for any $N \geq 2$ and $S := \bigcup_{N \geq 2} S_N$. The following lemmata indicate S is a scrambled set of the shift on $\mathbb{N}^{\mathbb{N}}$.

Lemma 4.2. For any $M, N \geq 2$ with $M \neq N$, then $S_M \cap S_N = \emptyset$.

Proof. Suppose that $M > N \geq 2$. Note that

$$S_M = \Delta_M(\Sigma_M) = \Delta_M(\{1, 2, \dots, M\}^{\mathbb{N}}),$$

and

$$S_N = \Delta_N(\Sigma_N) = \Delta_N(\{1, 2, \dots, N\}^{\mathbb{N}}).$$

By the definitions of Δ_M and Δ_N and $M > N$, the symbol M appears infinitely often in any $y \in S_M$, but it does not appear in any $x \in S_N$. Hence $S_M \cap S_N = \emptyset$. \square

Lemma 4.3. The set S is a scrambled set of the shift on $\mathbb{N}^{\mathbb{N}}$.

Proof. For any $u, v \in S \subset \mathbb{N}^{\mathbb{N}}$ with $u \neq v$, denoted by $u \in S_N$ and $v \in S_M$, we shall prove that (u, v) is a scrambled pair for the shift. The proof is divided into two cases according to $N = M$ or $N \neq M$.

(1). Assume that $u, v \in S_N = \Delta_N(\Sigma_N)$ for some $N \geq 2$, then there exist $x, y \in \Sigma_N$ with $x \neq y$ such that $u = \Delta_N(x)$ and $v = \Delta_N(y)$. Since $x \neq y$, there exists $k \geq 1$ such that $x_k \neq y_k$. By the definitions of g_N and Θ_N , the symbol x_k and y_k appear infinitely often in the same location of $\Theta_N \circ g_N(x)$ and $\Theta_N \circ g_N(y)$ respectively. By the definition of Δ_N , hence x_k and y_k appear infinitely often in the same

location of $\Delta_N(x)$ and $\Delta_N(y)$ respectively. That is to say, there exists an increasing sequence $\{n_i\}_{i \geq 1}$ such that

$$(\sigma^{n_i}(u))_1 = x_k \neq y_k = (\sigma^{n_i}(v))_1,$$

it follows that

$$\lim_{i \rightarrow \infty} d(\sigma^{n_i}(u), \sigma^{n_i}(v)) = 1 > 0.$$

On the other hand, according to the maps g_N and Θ_N , there exists an increasing sequence $\{m_j\}_{j \geq 1}$ such that

$$(\Theta_N \circ g_N(x))|_{m_j+1}^{m_j+j} = (\Theta_N \circ g_N(y))|_{m_j+1}^{m_j+j} = \underbrace{(1, 1, \dots, 1)}_j,$$

by the definition of Δ_N , there exists an increasing sequence $\{l_j\}_{j \geq 1}$ such that

$$u|_{l_j+1}^{l_j+j} = (\Delta_N(x))|_{l_j+1}^{l_j+j} = (\Theta_N \circ g_N(x))|_{m_j+1}^{m_j+j} = \underbrace{(1, 1, \dots, 1)}_j = (\Theta_N \circ g_N(y))|_{m_j+1}^{m_j+j} = (\Delta_N(y))|_{l_j+1}^{l_j+j} = v|_{l_j+1}^{l_j+j},$$

that is, $d(\sigma^{l_j}(u), \sigma^{l_j}(v)) \leq 2^{-j}$, which implies

$$\lim_{j \rightarrow \infty} d(\sigma^{l_j}(u), \sigma^{l_j}(v)) = 0.$$

(2). Suppose $u \in S_N$ and $v \in S_M$ such that $N, M \geq 2$ and $N \neq M$, then there exist $x \in \Sigma_N$ and $y \in \Sigma_M$ such that $u = \Delta_N(x)$ and $v = \Delta_M(y)$. By the definitions of g_N , Θ_N , g_M and Θ_M , we have

$$(\Theta_N \circ g_N(x))_1 = N \text{ and } (\Theta_M \circ g_M(y))_1 = M,$$

by the definition of Δ_N , there exists an increasing sequence $\{n_i\}_{i \geq 1}$ such that

$$u_{n_i+1} = (\Theta_N \circ g_N(x))_1 = N \text{ and } v_{n_i+1} = (\Theta_N \circ g_N(y))_1 = M,$$

it follows that

$$\lim_{i \rightarrow \infty} d(\sigma^{n_i}(u), \sigma^{n_i}(v)) = 1 > 0.$$

On the other hand, similarly to the case of $N = M$, there exists an increasing sequence $\{l_j\}_{j \geq 1}$ such that

$$u|_{l_j+1}^{l_j+j} = \underbrace{(1, 1, \dots, 1)}_j = v|_{l_j+1}^{l_j+j},$$

that is, $d(\sigma^{l_j}(u), \sigma^{l_j}(v)) \leq 2^{-j}$, which implies

$$\lim_{j \rightarrow \infty} d(\sigma^{l_j}(u), \sigma^{l_j}(v)) = 0.$$

□

Since the CF expansion of $x \in [0, 1) \cap \mathbb{Q}^c$ is unique, we can define a continuous bijection ϕ from $\mathbb{N}^{\mathbb{N}}$ to $[0, 1) \cap \mathbb{Q}^c$ by

$$\phi(a_1, a_2, a_3, \dots) = [a_1, a_2, a_3, \dots]$$

for $(a_1, a_2, a_3, \dots) \in \mathbb{N}^{\mathbb{N}}$. The following diagram is commutative, that is, $\phi \circ \sigma = T \circ \phi$.

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{\sigma} & \mathbb{N}^{\mathbb{N}} \\ \phi \downarrow & & \downarrow \phi \\ [0, 1) \cap \mathbb{Q}^c & \xrightarrow{T} & [0, 1) \cap \mathbb{Q}^c \end{array}$$

The following lemma implies that the set $\phi(S)$ is a scrambled set of T on $[0, 1)$.

Lemma 4.4. *The set $\phi(S)$ is a scrambled set of T on $[0, 1)$.*

Proof. For any $u, v \in S$ with $u \neq v$, denoted by $u \in S_N$ and $v \in S_M$, we shall prove that $(\phi(u), \phi(v))$ is a scrambled pair for T .

(1). **Lower limits.** According to Lemma 4.3, $\liminf_{n \rightarrow \infty} d(\sigma^n(u), \sigma^n(v)) = 0$, there exists an increasing sequence $\{n_j\}_{j \geq 1}$ such that

$$\lim_{j \rightarrow \infty} d(\sigma^{n_j}(u), \sigma^{n_j}(v)) = 0.$$

Since the map ϕ is continuous and $\phi \circ \sigma = T \circ \phi$, we obtain

$$\lim_{j \rightarrow \infty} |T^{n_j} \circ \phi(u) - T^{n_j} \circ \phi(v)| = \lim_{j \rightarrow \infty} |\phi(\sigma^{n_j}(u)) - \phi(\sigma^{n_j}(v))| = 0.$$

(2). **Upper limits.** The proof is divided into two cases according to $N = M$ or $N \neq M$.

(i) Suppose that $N = M$, that is, $u, v \in S_N$ for some $N \geq 2$, then $u_i, v_i \in \{1, 2, \dots, N\}$ for any $i \geq 1$. According to Lemma 4.3, $\limsup_{n \rightarrow \infty} d(\sigma^n(u), \sigma^n(v)) = 1$, there exists an increasing sequence $\{m_i\}_{i \geq 1}$ such that $(\sigma^{m_i}(u))_1 \neq (\sigma^{m_i}(v))_1$. Then

$$\begin{aligned} |T^{m_i} \circ \phi(u) - T^{m_i} \circ \phi(v)| &= |\phi(\sigma^{m_i}(u)) - \phi(\sigma^{m_i}(v))| \\ &\geq \min\{|I((\sigma^{m_i}(u))_1, (N+1))|, |I((\sigma^{m_i}(v))_1, (N+1))|\} \\ &\geq \min\{\lambda^{-1}|I((\sigma^{m_i}(u))_1)| |I(N+1)|, \lambda^{-1}|I((\sigma^{m_i}(v))_1)| |I(N+1)|\} \\ &\geq \lambda^{-1}|I(N+1)|^2 > 0 \end{aligned}$$

where the first inequality holds since the interval $I((\sigma^{m_i}(u))_1, (N+1))$ or $I((\sigma^{m_i}(v))_1, (N+1))$ is contained in the gap between the point $\phi(\sigma^{m_i}(u))$ and $\phi(\sigma^{m_i}(v))$ and $u_i, v_i \in \{1, 2, \dots, N\}$ for any $i \geq 1$ and the second inequality holds by Lemma 2.4. Therefore

$$\limsup_{n \rightarrow \infty} |T^n \circ \phi(u) - T^n \circ \phi(v)| > 0.$$

(ii) Suppose that $u \in S_N$, $v \in S_M$ for some $N, M \geq 2$ and $N \neq M$. According to (2) of Remark 4.1 and constructions of S_N and S_M , there exists an increasing sequence $\{l_i\}_{i \geq 1}$ such that

$$(\sigma^{l_i}(u))_1 (\sigma^{l_i}(u))_2 = N1, \quad (\sigma^{l_i}(v))_1 (\sigma^{l_i}(v))_2 = M1,$$

Then

$$\begin{aligned} |T^{l_i} \circ \phi(u) - T^{l_i} \circ \phi(v)| &= |\phi(\sigma^{l_i}(u)) - \phi(\sigma^{l_i}(v))| \\ &\geq \min\{|I(N, 2)|, |I(M, 2)|\} \\ &\geq \lambda^{-1}|I(2)| \min\{|I(N)|, |I(M)|\} > 0 \end{aligned}$$

where the first inequality holds since the interval $I(N, 2)$ or $I(M, 2)$ is contained in the gap between the point $\phi(\sigma^{l_i}(u))$ and $\phi(\sigma^{l_i}(v))$ and the second inequality holds by Lemma 2.4. Therefore

$$\limsup_{n \rightarrow \infty} |T^n \circ \phi(x) - T^n \circ \phi(y)| > 0.$$

□

4.2. Estimation of $\dim_{\mathbf{H}} \phi(\mathbf{S}_{\mathbf{N}})$. Recall that $E_N = \{x \in [0, 1) : 1 \leq a_n(x) \leq N, \forall n \geq 1\}$ for any $N \geq 2$. Since the Hausdorff dimension of rational number is zero, we only need to consider the set $E_N \cap \mathbb{Q}^c$. Note that $\phi(\{1, 2, \dots, N\}^{\mathbb{N}}) = E_N \cap \mathbb{Q}^c$. By the construction of S_N , $S_N \subset \{1, 2, \dots, N\}^{\mathbb{N}}$, we know that $\phi(S_N) \subset E_N \cap \mathbb{Q}^c$ for any $N \geq 2$, then

$$\dim_H \phi(S_N) \leq \dim_H (E_N \cap \mathbb{Q}^c) = \dim_H E_N.$$

Proposition 4.5. For any $N \geq 2$, we have $\dim_H \phi(S_N) = \dim_H E_N$.

Consider a map $g : \phi(S_N) \rightarrow E_N \cap \mathbb{Q}^c$ defined by

$$g(b) = \phi \circ \Delta_N^{-1} \circ \phi^{-1}(b)$$

for $b \in \phi(S_N)$. Since ϕ is continuous and bijective from $\mathbb{N}^{\mathbb{N}}$ to $[0, 1) \cap \mathbb{Q}^c$ and Δ_N is continuous and bijective from Σ_N to $S_N = \Delta_N(\Sigma_N)$, the map g is a continuous bijection on $\phi(S_N)$. Proposition 4.5 is the corollary of the following lemma, thus we omit the proof of Proposition 4.5.

Lemma 4.6. For any $\epsilon > 0$, the map g satisfies locally $\frac{1}{1+\epsilon}$ -Hölder condition.

Recall that the map $f : X \rightarrow \mathbb{R}$ ($X \subset \mathbb{R}$) satisfies locally α -Hölder condition, if there exist a real number $r > 0$ and a constant $C > 0$ such that, for any $x, y \in X$ with $|x - y| < r$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$

Before proving Lemma 4.6, we make use of a kind of symbolic space described as follows:

For a fixed integer $N \geq 2$ and any $n \geq 1$, we can check that

$$\phi(S_N) = \bigcap_{n \geq 1} \bigcup_{(b_1, b_2, \dots, b_n) \in A_n} I(b_1, b_2, \dots, b_n),$$

where

$$A_n = \{(b_1, b_2, \dots, b_n) \in \{1, 2, \dots, N\}^n : (b_1, b_2, \dots, b_n) = (x_1, x_2, \dots, x_n), \forall (x_1, x_2, \dots) \in S_N\}.$$

Recall that $R = \{r_n\}_{n \geq 1} := \bigcup_{m=1}^{\infty} \bigcup_{t=1}^m \{k : k = m^3 + t\}$. For any $n \geq 1$, let $t(n) = \#\{k \geq 1 : r_k \leq n, r_k \in R\}$, we obtain

$$t(n) \leq 1 + 2 + \dots + (n^{\frac{1}{3}} + 1) \leq \frac{(n^{\frac{1}{3}} + 1)(n^{\frac{1}{3}} + 2)}{2} \leq 2n^{\frac{2}{3}}, \text{ when } n \geq 8,$$

similarly,

$$t(n) \geq 1 + 2 + \dots + (n^{\frac{1}{3}} - 2) \geq \frac{(n^{\frac{1}{3}} - 2)(n^{\frac{1}{3}} - 1)}{2} \geq \frac{n^{\frac{2}{3}}}{8}, \text{ when } n \geq 64.$$

For any $(b_1, b_2, \dots, b_n) \in A_n$, let $\overline{(b_1, b_2, \dots, b_n)}$ be the finite word by eliminating the terms $\{b_{r_k} : r_k \leq n, r_k \in R\}$. Then

$$\overline{(b_1, b_2, \dots, b_n)} \in \{1, 2, \dots, N\}^{n-t(n)}.$$

For convenience, set

$$\overline{q_n}(b_1, b_2, \dots, b_n) = q_{n-t(n)}(\overline{(b_1, b_2, \dots, b_n)}) \text{ and } \bar{I}(b_1, b_2, \dots, b_n) = \bar{I}(\overline{(b_1, b_2, \dots, b_n)}).$$

For any $b = [b_1, b_2, \dots], c = [c_1, c_2, \dots] \in \phi(S_N) \subset [0, 1)$ and $b \neq c$, there exists $n \in \mathbb{N}$ such that $(b_1, \dots, b_n) = (c_1, \dots, c_n)$ and $b_{n+1} \neq c_{n+1}$.

Based on the following fact: for any $w \in \mathbb{N}^*$, the subintervals $I(w, i)$ ($i \in \mathbb{N}$) are positioned as follows (see FIGURE 1).

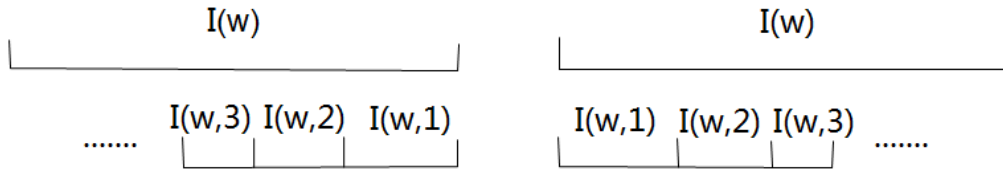


FIGURE 1. Subintervals $I(w, i)$ of $I(w)$ for the Gauss map

Lemma 4.7. We have $|b - c| \geq \lambda^{-2}|I(N + 1)|^2|I(b_1, \dots, b_n)|$ where λ is defined in Lemma 2.4.

Proof. Suppose that $b_{n+1} < c_{n+1}$. The interval $I(c_1, \dots, c_n, c_{n+1}, (N+1))$ is contained in the gap between b and c whether n is even or odd. Similarly to the case $c_{n+1} < b_{n+1}$, the interval $I(b_1, \dots, b_n, b_{n+1}, (N+1))$ is contained in the gap between b and c whether n is even or odd. By Lemma 2.4 and $1 \leq b_{n+1} \neq c_{n+1} \leq N$,

$$\begin{aligned} |b - c| &\geq \min\{|I(b_1, \dots, b_n, b_{n+1}, N + 1)|, |I(c_1, \dots, c_n, c_{n+1}, N + 1)|\} \\ &\geq \min\{\lambda^{-2}|I(b_1, \dots, b_n)| \cdot |I(b_{n+1})| \cdot |I(N + 1)|, \lambda^{-2}|I(c_1, \dots, c_n)| \cdot |I(c_{n+1})| \cdot |I(N + 1)|\} \\ &\geq \lambda^{-2}|I(b_1, \dots, b_n)||I(N + 1)|^2 \end{aligned}$$

□

Proof of Lemma 4.6: From Lemma 2.1, for any $(b_1, b_2, \dots, b_n) \in A_n$, we have

$$\overline{q_n}^2(b_1, b_2, \dots, b_n) = q_{n-t(n)}^2(\overline{b_1, b_2, \dots, b_n}) \geq 2^{(n-t(n)-1)}.$$

Let $\epsilon > 0$, there exists $K = K(\epsilon) > 64$ such that for any $n \geq K$, we have

$$2^{(n-t(n)-1)\epsilon} \geq 2 \cdot (N+1)^{2t(n)}.$$

By Lemma 2.1 and Lemma 2.2,

$$\begin{aligned} |I(b_1, b_2, \dots, b_n)| &\geq \frac{1}{2q_n^2(b_1, b_2, \dots, b_n)} \geq \frac{1}{2q_{n-t(n)}^2(\overline{b_1, b_2, \dots, b_n})(N+1)^{2t(n)}} \\ &\geq \frac{1}{q_{n-t(n)}^{2(1+\epsilon)}(\overline{b_1, b_2, \dots, b_n})} \geq |\overline{I}(b_1, b_2, \dots, b_n)|^{1+\epsilon} \end{aligned}$$

Let $r < \lambda^{-2}|I(N+1)|^2 \cdot \min_{(b_1, \dots, b_K) \in A_K} |I(b_1, \dots, b_K)|$. For any $c \in (b-r, b+r)$, there exists n such that $(b_1, \dots, b_n) = (c_1, \dots, c_n)$ and $b_{n+1} \neq c_{n+1}$, where $b = [b_1, b_2, \dots]$, $c = [c_1, c_2, \dots] \in \phi(S_N) \subset [0, 1]$, then

$$\begin{aligned} |g(b) - g(c)| &= |\phi \circ \Delta_N^{-1} \circ \phi^{-1}(b) - \phi \circ \Delta_N^{-1} \circ \phi^{-1}(c)| \\ &\leq |\overline{I}(b_1, b_2, \dots, b_n)| \leq |I(b_1, b_2, \dots, b_n)|^{\frac{1}{1+\epsilon}} \\ &\leq (\lambda^2 |I(N+1)|^{-2})^{\frac{1}{1+\epsilon}} |b - c|^{\frac{1}{1+\epsilon}} \end{aligned}$$

where the last inequality holds by Lemma 4.7. \square

Proof of Theorem 1.3: By Lemma 4.4, the set $\phi(S)$ is a scrambled set of T on $[0, 1]$ where $S = \bigcup_{N \geq 2} S_N$.

Since $\dim_H \phi(S) \geq \dim_H \phi(S_N) = \dim_H E_N \geq 1 - \frac{1}{N \log 2}$ for any $N \geq 2$, thus $\dim_H \phi(S) = 1$. \square

Acknowledgement. The authors thank Professor Ying Xiong for the helpful suggestions. The work was supported by NSFC 11371148, Guangdong Natural Science Foundation 2014A030313230, and "Fundamental Research Funds for the Central Universities" SCUT 2015ZZ055 and 2015ZZ127.

REFERENCES

1. F. Blanchard, E. Glasner and S. Kolyada, On Li-Yorke pairs. *J. Reine. Angew. Math.*, (2002), 547:51-68.
2. H. Bruin and V. Jiménez López, On the Lebesgue measure of Li-Yorke pairs for interval maps. *Comm. Math. Phys.* 299 (2010), no. 2, 523-560.
3. C. Faivre, A central limit theorem related to decimal and continued fraction expansion, *Arch. Math.* 70(6) (1998) 455-463.
4. K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*. New York: Wiley, 1990.
5. I. J. Good, The fractional dimensional theory of continued fractions, *Proc. Cambridge Philos. Soc.* 37 (1941), 199-228.
6. H. Hu and Y.-L. Yu, On Schmidt's game and the set of points with non-dense orbits under a class of expanding maps, *J. Math. Anal. Appl.* 418(2) (2014), 906-920.
7. I. Jarník, Zur metrischen theorie der diophantischen approximationen. *Prace Mat. Fiz.* 36 (1928), 91-106.
8. I. Kan, A chaotic function possessing a scrambled set with positive Lebesgue measure. *Proc. Amer. Math. Soc.* 92 (1984), 45-49.
9. A. Ya. Khintchine, *Continued Fractions*. University of Chicago Press, 1964.
10. D.Y. Kleinbock, Nondense orbits of flows on homogeneous spaces, *Ergodic Theory Dynam. Systems* 18 (2) (1998) 373-396.
11. M. Kuchta and J. Smítal, Two point scrambled set implies chaos, *European Conference on Iteration Theory (ECIT 87)*, World Sci. Publishing, Singapore, 1989, 427-430.
12. B. Li and Y.-C. Chen, Chaotic and topological properties of β -transformations. *J. Math. Anal. Appl.* 383 (2011), 585-596.
13. B. Li, B.-W. Wang, J. Wu, and J. Xu, The shrinking target problem in the dynamical system of continued fractions. *Proc. Lond. Math. Soc.* (3) 108 (2014), no. 1, 159-186.
14. T. Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly* 82 (1975), 985-992.
15. D. Mayer, On the thermodynamic formalism for the Gauss map, *Comm. Math. Phys.* 130 (1990) 311-333.
16. I. Mizera, Continuous chaotic functions of an interval have generically small scrambled sets. *Bull. Austral. Math. Soc.* 37 (1988), no. 1, 89-92.
17. J. Neunhäuserer, Li-Yorke pairs of full Hausdorff dimension for some chaotic dynamical systems. *Math. Bohem.* 135 (2010), no. 3, 279-289.
18. W. Philipp, Some metrical theorems in number theory II, *Duke Math. J.* 37 (1970) 447-458.
19. J. Smítal, A chaotic function with some extremal properties. *Proc. Amer. Math. Soc.* 87 (1983), 54-56.
20. M. Urbański, The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system, *Nonlinearity* 4 (2) (1991) 385-397.

21. B.-W. Wang and J. Wu, Hausdorff dimension of certain sets arising in continued fraction expansions. *Adv. Math.* 218 (2008), no. 5, 1319-1339.
22. H. Wang and J. Xiong, Chaos for subshifts of finite type. *Acta Math. Sin. (Engl. Ser.)* 21 (2005), no. 6, 1407-1414.
23. H. Wang and J. Xiong, The Hausdorff dimension of chaotic sets for interval self-maps. *Topology Appl.* 153 (2006), no. 12, 2096-2103.
24. J. Wu, A remark on the growth of the denominators of convergents. *Monatsh. Math.* 147(3)(2006), 259-264.
25. J. Xiong, Hausdorff dimension of a chaotic set of shift of a symbolic space. *Science in China (Series A)*, 38(1995), 6:696-708.
26. J. Xu, The distribution of the largest digit in continued fraction expansions. *Math. Proc. Camb. Phil. Soc.* 146 (2009), 207-212.

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